Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet for 2021-10-13

Conceptual questions

Question 1. Back in Chapter 10 you saw the formula

$$\int_{\theta_0}^{\theta_1} \frac{1}{2} r^2 \, \mathrm{d}\theta$$

for computing polar areas. Explain how this formula is a special case of the material in §15.3.

Computations

Problem 1. Consider the two regions

(a) $x^2 + y^2 \le z \le 2x$
(b) $(x+1)^2 + y^2 \le z \le 2(x+1)$

How are they related? Set up integrals which evaluate the volume of each, and verify that they are equal.

Problem 2 (Stewart 15.3.40)**.** Here is a very famous application of polar coordinates.

(a) Evaluate the expression

$$\lim_{a\to\infty} \iint_{D_a} e^{-x^2-y^2} \, \mathrm{d}x \, \mathrm{d}y$$

where D_a is the disk of radius a centered at $(0,0)$. That is, evaluate the integral in terms of a, and then compute the limit of the resulting expression.

(b) Now consider the limit

$$\lim_{b\to\infty} \iint_{S_b} e^{-x^2-y^2} \, \mathrm{d}x \, \mathrm{d}y$$

where S_b is the square with opposite corners $(-b,-b)$ and (b,b). Explain why this limit also exists and is equal to the answer from (a). Hint: Do not try to compute the integral in terms of b; it's impossible. Can you use the Squeeze Theorem somehow?

(c) The function e^{-x^2} has no elementary antiderivative. Despite this, it is possible to compute the integral

$$I = \int_{-\infty}^{\infty} e^{-x^2} \, \mathrm{d}x = \lim_{a\to\infty} \int_{-a}^{a} e^{-x^2} \, \mathrm{d}x.$$

Deduce the value of I from your answer to (b).

(d) The *standard normal distribution* is the probability distribution with density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

Verify that this is a valid density function (i.e., that it is always nonnegative and that it integrates to 1).